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VORTEX MOTIONS OF LIQUID IN A NARROW CHANNEL

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Quasi-linear integrodifferential equations that describe vortex flows of an ideal incompressible liquid in a narrow curved channel in the Eulerian–Lagrangian coordinate system are considered. The necessary and sufficient conditions for hyperbolicity of the system of equations of motion are obtained for flows with a monotonic velocity depth profile. The propagation velocities of the characteristics and the characteristic form of the system are calculated. A particular solution is given in which the system of integrodifferential equations changes type with time. The solution of the Cauchy problem is given for linearized equations. An example of initial data for which the Cauchy problem is ill-posed is constructed.

1. Derivation of the Equations of Motion. The solution of the boundary-value problem

$$\begin{gathered} u_T + uu_X + vu_Y + p_X = 0, \qquad \varepsilon^2(v_T + uv_X + vv_Y) + p_Y = -1, \\ u_X + v_Y = 0, \qquad -\infty < X < \infty, \qquad 0 \leqslant Y \leqslant h(X), \\ v(T,X,0) = 0, \qquad u(T,X,h)h_X = v(T,X,h) \end{gathered} \tag{1.1}$$

describes the plane-parallel motions of a layer of an ideal incompressible liquid bounded by a solid wall $Y = h(X)$ and a level floor in a gravitational field. The variables $\bar u = (gH_0)^{1/2}u$, $\bar v = (gH_0)^{1/2}H_0L_0^{-1}v$, $\bar p = \rho g H_0 p$, $\bar T = L_0(gH_0)^{-1/2}T$, $\bar X = L_0X$, and $\bar Y = H_0Y$ are the dimensional components of the velocity vector, the pressure, the time, and the Cartesian coordinates, respectively; u, v, p, T, X, and Y are the dimensionless quantities corresponding to them. The parameters H_0 and L_0 determine the characteristic vertical and horizontal scales, ρ is the density, and g is the acceleration of gravity. In a narrow-channel approximation, the parameter $\varepsilon = H_0L_0^{-1}$ is assumed to be small, and terms of order ε^2 in Eqs. (1.1) are ignored, which enables us to represent the pressure in the form $p(T,X,Y) = h(X) - Y + p^*(T,X)$, where p^* is the dimensionless pressure at the upper boundary of the channel. Integration of the continuity equation yields the equation

$$v = -\int_0^Y u_X\,dY.$$

After transformations, we have the problem of finding u and p^*:

$$u_T + uu_X + vu_Y + h_X + p^*_X = 0, \qquad \Big(\int_0^h u\,dY\Big)_X = 0 \tag{1.2}$$

(the functions p and v were defined above). In this model, the absence of vorticity is equivalent to the condition $u_Y = 0$. We consider vortex flows with a monotonic velocity depth profile ($u_Y > 0$).

We pass to the Eulerian–Lagrangian coordinates x and λ [1],

$$T = t, \qquad X = x, \qquad Y = \Phi(t,x,\lambda) \quad (0 \leqslant \lambda \leqslant 1), \tag{1.3}$$

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where the function $\Phi(t,x,\lambda)$ is the solution of the Cauchy problem

$$\Phi_t + u(t,x,\Phi)\Phi_x = v(t,x,\Phi), \qquad \Phi(0,x,\lambda) = \lambda h(x). \tag{1.4}$$

By virtue of (1.2) and (1.3), to define the functions $u(t,x,\lambda)$ and $H(t,x,\lambda) = \Phi_\lambda$ we obtain

$$u_t + uu_x - (u_{1t} + u_1 u_{1x}) = 0, \qquad H_t + (uH)_x = 0, \qquad \int_0^1 H\,d\lambda = h(x) \tag{1.5}$$

[$u_1 = u(t,x,1)$ and $p_x = -(u_{1t} + u_1 u_{1x})$, and since p_x does not depend on λ, this function can be expressed in terms of the velocity u and its derivatives for a fixed λ, such as $\lambda = 1$]. The change of variables (1.3) is reversible under the condition $\Phi_\lambda \neq 0$. We take $\Phi_\lambda > 0$.

For further transformation of the equations of motion, we use the equation

$$\int_0^1 uH\,d\lambda = Q(t), \tag{1.6}$$

which means that at each time the liquid flow rate $Q(t)$ in the channel does not depend on the cross section. Let the flow rate $Q(t)$ be given and let u_λ and $\theta = H/u_\lambda$ (the quantity inversely proportional to the vorticity) be the desired functions. Then we can pass from Eqs. (1.5) to the system

$$u_{\lambda t} + uu_{\lambda x} + u_\lambda u_x = 0, \qquad \theta_t + u\theta_x = 0, \tag{1.7}$$

where the functions u and u_x are expressed, in accordance with (1.6), by the equations

$$\begin{gathered} u = u_1 - \int_\lambda^1 u_\nu\,d\nu, \qquad u_1 = \Big(\int_0^1 u_\lambda\theta\,d\lambda\Big)^{-1}\Big[Q(t) + \int_0^1 u_\lambda\Big(\int_0^\lambda u_\nu\theta\,d\nu\Big)\,d\lambda\Big], \\ u_x = -\int_\lambda^1 u_{\nu x}\,d\nu + \Big(\int_0^1 u_\lambda\theta\,d\lambda\Big)^{-1}\Big[-u_1\Big(\int_0^1 u_{\lambda x}\theta\,d\lambda + \int_0^1 u_\lambda\theta_x\,d\lambda\Big) \\ + \int_0^1 u_{\lambda x}\Big(\int_0^\lambda u_\nu\theta\,d\nu\Big)\,d\lambda + \int_0^1 u_\lambda\Big(\int_0^\lambda u_{\nu x}\theta\,d\nu\Big)\,d\lambda + \int_0^1 u_\lambda\Big(\int_0^\lambda u_\nu\theta_x\,d\nu\Big)\,d\lambda\Big]. \end{gathered} \tag{1.8}$$

If the functions u_λ and θ are determined, we know $H = \theta u_\lambda$, the upper boundary $h = \int_0^1 u_\lambda\theta\,d\lambda$ of the channel, and, by virtue of (1.8), u. The equation $h_t = 0$, which expresses the fact that the upper boundary is fixed, is a consequence of the equations, since we have

$$h_t = \int_0^1 (\theta_t u_\lambda + \theta u_{\lambda t})\,d\lambda = -\int_0^1 (uu_\lambda\theta_x + uu_{\lambda x}\theta + u_\lambda u_x\theta)\,d\lambda = -[Q(t)]_x = 0.$$

The equations $p_x = h_x + p_x^* = -(u_{1t} + u_1 u_{1x})$, $\Phi_\lambda = H$, $\Phi(t,x,0) = 0$, and (1.4) enable us to find the pressure (to within an arbitrary function of t), Φ, and the vertical velocity component. System (1.1) in the narrow-channel approximation is thus reduced to problem (1.7), to which, in contrast to Eqs. (1.5), methods of studying of the hyperbolicity [2] can be applied.

Remark 1. The results obtained also hold for three-dimensional axisymmetric flows. In this case, the system of equations of motion, in the absence of an external force field and in a long-wavelength approximation, has the form

$$\begin{gathered} u_t + uu_x + wu_r + p_x = 0, \qquad p_r = 0, \\ u_x + w_r + r^{-1}w = 0, \qquad (uR_x - w)\Big|_{r=R} = 0 \quad (0 \leqslant r \leqslant R(x)). \end{gathered}$$

The change of the variables $y = r^2/2$, $v = rw$, and $h(x) = R^2(x)/2$ reduces this system to equations similar to (1.1) for $\varepsilon = 0$.

2. Hyperbolicity Conditions of the Equations of Motion. We shall formulate the necessary and sufficient conditions for system (1.7) to be hyperbolic. Analysis of the characteristics of system (1.7) is based on a generalization of the concept of hyperbolicity for systems of equations with operator coefficients, proposed in [2] and used in [3] to investigate the problem with a free boundary.

System (1.7) can be written

$$\mathbf{U}_t + A\mathbf{U}_x = 0, \tag{2.1}$$

where $\mathbf{U} = (u_\lambda(t,x,\lambda), \theta(t,x,\lambda))^t$, and A is a matrix with operator coefficients that arises from the substitution of Eqs. (1.8) into Eq. (1.7), and it operates on a vector function $\mathbf{f}$ by the rule

$$A\mathbf{f} = \Big(uf_1 - u_\lambda \int_\lambda^1 f_1\,d\nu + u_\lambda h^{-1}\Big[-u_1\Big(\int_0^1 f_1\theta\,d\lambda + \int_0^1 u_\lambda f_2\,d\lambda\Big) + \int_0^1 f_1\Big(\int_0^\lambda u_\nu\theta\,d\nu\Big)\,d\lambda$$
$$+ \int_0^1 u_\lambda\Big(\int_0^\lambda f_1\theta\,d\nu\Big)\,d\lambda + \int_0^1 u_\lambda\Big(\int_0^\lambda u_\nu f_2\,d\nu\Big)\,d\lambda\Big],\ uf_2\Big)^t.$$

The characteristic of system (2.1) is determined by the differential equation $x'(t) = k(t,x)$, where k is the eigenvalue of the operator A^* (the propagation velocity of the characteristic). The solution of the equation

$$(\mathbf{F}, (A - kI)\varphi) = 0 \tag{2.2}$$

for the vector functional $\mathbf{F} = (F_1, F_2)$, which operates on the arbitrary, infinitely differentiable vector function $\varphi = (\varphi_1, \varphi_2)^t$ of the variable λ (the dependence on t and x as parameters), is sought in the class of locally integrable or generalized functions. The expression $(\mathbf{F}, \varphi)$ denotes the result of the action of the functional $\mathbf{F}$ on the test vector function. We assume the functions u_λ and θ to be infinitely differentiable with respect to λ. The action of $\mathbf{F}$ on Eq. (2.1) yields the equation for the characteristic

$$(\mathbf{F}, \mathbf{U}_t + k\mathbf{U}_x) = 0. \tag{2.3}$$

System (2.1) is hyperbolic if all the eigenvalues k are real and the set of equations for the characteristics (2.3) is equivalent to Eqs. (2.1).

With allowance for Eq. (2.2) and the nondependence of the test functions φ_1 and φ_2, we obtain the equations

$$\Big(F_1, (u-k)\varphi_1 - u_\lambda \int_0^1 \varphi_1\,d\nu + u_\lambda h^{-1}\Big[-u_1\int_0^1 \varphi_1\theta\,d\nu \int_0^1 \varphi_1\Big(\int_0^\lambda u_\nu\theta\,d\nu\Big)\,d\lambda + \int_0^1 u_\lambda\Big(\int_0^\lambda \varphi_1\theta\,d\nu\Big)\,d\lambda\Big]\Big) = 0; \tag{2.4}$$

$$(F_2, (u-k)\varphi_2) + h^{-1}\Big[-u_1\int_0^1 u_\lambda\varphi_2\,d\lambda + \int_0^1 u_\lambda\Big(\int_0^\lambda u_\nu\varphi_2\,d\nu\Big)\,d\lambda\Big](F_1, u_\lambda) = 0. \tag{2.5}$$

We consider the set of eigenvalues k, which belong to the complex plane except for the interval $[u_0, u_1]$. With the use of the function $\psi(\lambda) = -\int_\lambda^1 \varphi_1(\nu)\,d\nu$, Eq. (2.4) takes the form

$$\Big(F_1, \Big[\Big(u(\lambda) - k\Big)\Big(\psi(\lambda) + h^{-1}\Big(u_0\theta_0\psi_0 + \int_0^1 u\theta_\nu\psi\,d\nu\Big)\Big)\Big]_\lambda\Big) = 0. \tag{2.4'}$$

Here and below, the 0 and 1 mean that the functions u, θ, and ψ are taken for $\lambda = 0$ and 1. We represent the functional F_1 as a combination of V and W, where $(V, \varphi) = -\int_\lambda^1 \varphi(\nu)\,d\nu$. To determine W, we obtain the equation

$$(W,(u(\lambda)-k)\psi(\lambda))+h^{-1}\Big(u_0\theta_0\psi_0+\int\limits_0^1\psi u\theta_\nu\,d\nu\Big)(W,(u(\lambda)-u_1))=0, \tag{2.6}$$

from which we find $(W,\varphi)=\int\limits_0^1\theta(\lambda)[u(\lambda)\varphi(\lambda)(u(\lambda)-k)^{-1}]_\lambda\,d\lambda$. Hence the functional F_1 acts by the rule

$$(F_1,\varphi)=\int\limits_0^1\theta(\lambda)\Big[u(\lambda)(u(\lambda)-k)^{-1}\int\limits_\lambda^1\varphi(\nu)\,d\nu\Big]_\lambda\,d\lambda. \tag{2.7}$$

Taking $\psi=(u(\lambda)-u_1)(u(\lambda)-k)^{-1}$ in (2.6), after some transformations we obtain the characteristic equation

$$k(k-u_1)\int\limits_0^1\theta(\lambda)u_\lambda(u(\lambda)-k)^{-2}\,d\lambda=0, \tag{2.8}$$

which determines the discrete spectrum of the operator A^*. We note that if u does not vanish, Eq. (2.8) has a single real root $k=0$, since we have $\theta u_\lambda=H=\Phi_\lambda>0$ and $k\neq u$. Other characteristic roots, if they exist for this solution, are complex. Substituting $k=0$ into (2.5) and (2.7), we find the functional corresponding to this eigenvalue:

$$(F_1^0,\varphi)=\int\limits_0^1\theta(\lambda)\varphi(\lambda)\,d\lambda,\qquad (F_2^0,\varphi)=\int\limits_0^1 u_\lambda\varphi(\lambda)\,d\lambda.$$

Below, we shall show that the operator A^* has a continuous characteristic spectrum $k^\lambda(t,x)=u(t,x,\lambda)$, and we find the corresponding eigenfunctionals. Proceeding by analogy with the foregoing, we represent F_1 as a composition $W\circ V$. We specify the action of the functional V by the rule $(V,\varphi)=(u(\nu)-u(\lambda))^{-1}\int\limits_\lambda^\nu\varphi(\mu)\,d\mu$. To determine W, we obtain, in accordance with (2.4′), the equation

$$\Big(W,\psi(\nu)+h^{-1}\Big(u_0\theta_0\psi_0+\int\limits_0^1 u\theta_\mu\psi\,d\mu\Big)\Big)=0,$$

from which we obtain $(W,\varphi)=\int\limits_0^1\theta[u\varphi]_\nu\,d\nu$. Finally, the action of the functional $F_1^{1\lambda}=W\circ V$ is determined to be

$$(F_1^{1\lambda},\varphi)=\int\limits_0^1\theta(\nu)\Big[(u(\nu)-u(\lambda))^{-1}u(\nu)\int\limits_\lambda^\nu\varphi(\mu)\,d\mu\Big]_\nu\,d\nu.$$

From Eq. (2.5) we find the functional $F_2^{1\lambda}$:

$$(F_2^{1\lambda},\varphi)=\int\limits_0^1 u(\nu)u_\nu\varphi(\nu)(u(\nu)-u(\lambda))^{-1}\,d\nu.$$

It is obvious that Eqs. (2.4) and (2.5) have one other nontrivial solution: $(F_1^{2\lambda},\varphi)=0$, $(F_2^{2\lambda},\varphi)=\delta(\nu-\lambda)$.

To obtain equations for the characteristics with $u\neq 0$, we act on Eqs. (1.7) by the functionals $\mathbf{F}^{1\lambda}$, $\mathbf{F}^{2\lambda}$, and $\mathbf{F}^0$. The action of $\mathbf{F}^{1\lambda}$ on the system yields the equation

$$\int\limits_0^1\theta(\nu)[u(\nu)(u_t(\nu)+u(\nu)u_x(\nu)-u_t(\lambda)-u(\lambda)u_x(\lambda))(u(\nu)-u(\lambda))^{-1}]_\nu\,d\nu$$
$$+\int\limits_0^1 u(\nu)u_\nu(\theta_t(\nu)+u(\nu)\theta_x(\nu))(u(\nu)-u(\lambda))^{-1}\,d\nu=0,$$

which after transformations reduces to the equation

$$u(\lambda)\Big[\int_0^1 [u_\nu(\theta_t(\nu)+u(\lambda)\theta_x(\nu))+\theta(\nu)(u_{\nu t}+u(\lambda)u_{\nu x})](u(\nu)-u(\lambda))^{-1}\,d\nu$$

$$-\int_0^1 \theta(\nu)u_\nu[u_t(\nu)+u(\lambda)u_x(\nu)-u_t(\lambda)-u(\lambda)u_x(\lambda)](u(\nu)-u(\lambda))^{-2}\,d\nu\Big]$$

$$+\int_0^1 u_\nu(\theta_t(\nu)+u(\lambda)\theta_x(\nu))\,d\nu+\int_0^1 \theta(\nu)(u_{\nu t}+u(\lambda)u_{\nu x})\,d\nu=0.$$

With the use of the functions h and $R(\lambda)=\int_0^1 u_\nu\theta(\nu)(u(\nu)-u(\lambda))^{-1}\,d\nu$, the system of equations for these characteristics takes the form

$$u(R_t+uR_x)+h_t+uh_x=0,\qquad \theta_t+u\theta_x=0,\qquad h_t=0. \tag{2.9}$$

Remark 2. If u vanishes $[u(\lambda_*)=0]$, the functionals $\mathbf{F}^{1\lambda_*}$ and $\mathbf{F}^0$ coincide, and the first equation of (2.9) with $u=0$ is satisfied automatically. In this case, we use the eigenfunctionals $\mathbf{F}^0$ and $\mathbf{F}^{2\lambda}$ and the associated one $\mathbf{P}^{1\lambda}=(P_1^{1\lambda},P_2^{1\lambda})$, which acts on the test function φ by the rule

$$(P_1^{1\lambda},\varphi)=\int_0^1 \theta(\nu)\Big[(u(\nu)-u(\lambda))^{-1}\int_\lambda^\nu \varphi(\mu)\,d\mu\Big]_\nu\,d\nu,$$

$$(P_2^{1\lambda},\varphi)=\int_0^1 u_\nu\varphi(\nu)(u(\nu)-u(\lambda))^{-1}\,d\nu$$

and has the property $(\mathbf{P}^{1\lambda},(A-uI)\varphi)=(\mathbf{F}^0,\varphi)$. By acting on system (1.7) by the functionals $\mathbf{F}^0$, $\mathbf{P}^{1\lambda}$, and $\mathbf{F}^{2\lambda}$, we obtain the equations

$$R_t+uR_x+h_x=0,\qquad \theta_t+u\theta_x=0,\qquad h_t=0, \tag{2.9$'$}$$

which are equivalent to Eqs. (2.9) for the characteristics with $u\neq 0$.

The hyperbolicity conditions for system (1.7) are formulated in terms of the complex function $\chi(z)=\int_0^1 \theta u_\lambda(u-z)^{-2}\,d\lambda$ or, to be more precise, its limiting values

$$\chi^\pm(u(\lambda))=-\theta_1(u_1-u(\lambda))^{-1}+\theta_0(u_0-u(\lambda))^{-1}+\int_0^1 \theta_\nu(u(\nu)-u(\lambda))^{-1}\,d\nu\pm\pi i\theta_\lambda/u_\lambda$$

from the upper and lower half-planes on the interval $[u_0,u_1]$.

Lemma 1. *For the solution u_λ and θ, Eq.* (2.8) *has no complex roots if the following condition is satisfied:*

$$æ=\Delta\arg\,(\chi^+/\chi^-)=0,\qquad \chi^\pm\neq 0 \tag{2.10}$$

(*the increment of the argument upon variation of λ from zero to unity*).

Proof. In Eq. (2.8) only the integral cofactor, which coincides with $\chi(k)$, can have complex roots. It is therefore sufficient to check the lemma's statement for the equation $\chi(k)=0$. We draw a contour γ of the dumbbell type around the interval of variation of the function u and draw a circle Γ of a sufficiently large radius [such that all the roots of the equation $\chi(k)=0$ lie inside the circle] (Fig. 1). In the domain D (the intersection of the exterior of the dumbbell and the circle), the function $\chi(k)$ is analytical and has no poles. By virtue of the principle of the argument, the number of zeros of $\chi(k)$ in this domain equals the

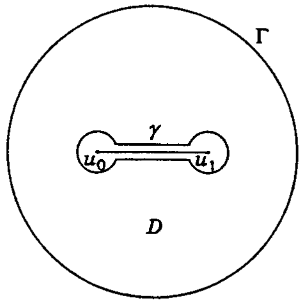

Fig. 1

increment of the argument along the contour $\gamma \cup \Gamma$ divided by 2π. The increment in the argument upon going counterclockwise around the circle Γ is -2π (a second-order zero). In going clockwise around each small open circle confining the points u_0 and u_1, the argument obtains an increment of π each (a first-order pole), which adds up exactly, to within the sign, to the increment obtained in going around the contour Γ. The number of zeros of the function χ in the domain D therefore equals the increment of the argument on the handle of the dumbbell divided by 2π. It follows from this that condition (2.10) is necessary and sufficient for the absence of complex zeros of the equation $\chi(k) = 0$. Lemma 1 is proved.

The requirement $\chi^{\pm} \neq 0$ excludes the neutral case. Therefore, if (2.10) is satisfied, there are no complex characteristic roots not only for the given solution but also for sufficiently small perturbations of it.

The next lemma establishes the conditions under which Eqs. (2.9) for the characteristics are equivalent to Eqs. (1.7).

Lemma 2. *Let the components of the vector function* $\mathbf{S} = (S_1, S_2)^t$ *be such that* $\int_\lambda^1 S_1\, d\nu$ *satisfies the Hölder condition and while* S_2 *is continuous in the variable* λ, *and the equations* $(\mathbf{F}^{1\lambda}, \mathbf{S}) = 0$, $(\mathbf{F}^{2\lambda}, \mathbf{S}) = 0$, *and* $(\mathbf{F}^0, \mathbf{S}) = 0$ *and condition* (2.10) *are satisfied. Then* $\mathbf{S}(\lambda) \equiv \mathbf{0}$, *where* $0 \leqslant \lambda \leqslant 1$.

Proof. From the equation $(\mathbf{F}^{2\lambda}, \mathbf{S}) = 0$, it follows that the component S_2 equals zero. Using this, to determine the function S_1 we obtain

$$\int_0^1 \theta(\nu)\left[(u(\nu) - u(\lambda))^{-1} u(\nu) \int_\lambda^\nu S_1(\mu)\, d\mu\right]_\nu d\nu = 0, \quad \int_0^1 \theta(\lambda) S_1(\lambda)\, d\lambda = 0. \tag{2.11}$$

By integration by parts and the substitution $\psi(\lambda) = -\int_\lambda^1 S_1(\mu)\, d\mu$, we transform Eqs. (2.11) to the form

$$u_1\theta_1(u_1 - u(\lambda))^{-1}\psi(\lambda) + u_0\theta_0(u_0 - u(\lambda))^{-1}(\psi_0 - \psi(\lambda)) + \int_0^1 u(\nu)\theta_\nu(u(\nu) - u(\lambda))^{-1}(\psi(\nu) - \psi(\lambda))\, d\nu = 0,$$

$$\psi_0\theta_0 + \int_0^1 \theta_\lambda \psi(\lambda)\, d\lambda = 0.$$

Eliminating ψ_0, to determine the function ψ we write the singular integral equation

$$\left[u_1\theta_1(u_1 - u(\lambda))^{-1} - u_0\theta_0(u_0 - u(\lambda))^{-1} - \int_0^1 u(\nu)\theta_\nu(u(\nu) - u(\lambda))^{-1}\, d\nu\right]\psi(\lambda)$$

$$-u_0(u_0 - u(\lambda))^{-1}\int_0^1 \theta_\nu\psi(\nu)\, d\nu + \int_0^1 u(\nu)\theta_\nu(u(\nu) - u(\lambda))^{-1}\psi(\nu)\, d\nu = 0,$$

which we reduce by simple transformations to the form

$$u(\lambda)\Big[(-\theta_1(u_1-u(\lambda))^{-1}+\theta_0(u_0-u(\lambda))^{-1}+\int_0^1\theta_\nu(u(\nu)-u(\lambda))^{-1}\,d\nu)\psi(\lambda)$$

$$-(u(\lambda)-u_0)^{-1}\int_0^1\theta_\nu(u(\nu)-u_0)(u(\nu)-u(\lambda))^{-1}\psi(\nu)\,d\nu\Big]=0.$$

By virtue of Remark 2, the factor $u(\lambda)$ in this expression can be cancelled. The change of the variable $\xi=u(\nu)$ [$\xi_0=u(0)$, $\xi_1=u(1)$, and $z=u(\lambda)$] reduces this equation to the singular integral equation, which is similar to the characteristic equation [4],

$$a(z)\tilde{\psi}-(\pi i)^{-1}\int_{\xi_0}^{\xi_1}b(\xi)(\xi-z)^{-1}\tilde{\psi}(\xi)\,d\xi=0, \tag{2.12}$$

where the functions $a(z)$ and $b(\xi)$ are given by the equations

$$a(z)=(z-\xi_0)(-\tilde{\theta}_1(\xi_1-z)^{-1}+\tilde{\theta}_0(\xi_0-z)^{-1}+\int_{\xi_0}^{\xi_1}\tilde{\theta}_\xi(\xi-z)^{-1}\,d\xi),$$

$$b(\xi)=\pi i\tilde{\theta}_\xi(\xi-\xi_0)\qquad (f(\lambda)=\tilde{f}(z)).$$

We introduce a piecewise-holomorphic function that vanishes at infinity:

$$\Psi(z)=(2\pi i)^{-1}\int_{\xi_0}^{\xi_1}b(\xi)\tilde{\psi}(\xi)(\xi-z)^{-1}\,d\xi.$$

From the Sokhotskii–Plemelj equations we obtain the equalities

$$a(z)\tilde{\psi}(z)=\Psi^+(z)+\Psi^-(z),\qquad b(z)\tilde{\psi}(z)=\Psi^+(z)-\Psi^-(z),$$

from which it is seen that the solution of Eq. (2.12) can be reduced to the solution of the homogeneous conjugation problem $\Psi^+(z)=G(z)\Psi^-(z)$, where $G(z)=(a(z)+b(z))(a(z)-b(z))^{-1}$ is a given function while Ψ is a function to be found. From the known Ψ, it is easy to find $\tilde{\psi}=(2(a+b))^{-1}\Psi^+$. We note that we have $\tilde{\chi}^\pm(z)=(a(z)\pm b(z))(z-\xi_0)^{-1}$. As z varies from ξ_0 to ξ_1, the increment of the argument of the function $G(z)$ therefore equals the increment of the argument $\chi^+(z)(\chi^-(z))^{-1}$, and it equals zero from condition (2.10). The index of the conjugation problem therefore equals zero. In this case, according to [4], a homogeneous conjugation problem has only a trivial solution in the class of functions that vanish at infinity, and hence $\tilde{\psi}(z)=\psi(\lambda)=0$. Since we have $S_1(\lambda)=\psi'(\lambda)$, we obtain $S_1(\lambda)\equiv 0$. Lemma 2 is proved.

Theorem. *For flows having a monotonic velocity depth profile, conditions* (2.10) *are necessary and sufficient for Eqs. (1.7) to be hyperbolic if the functions u and θ are differentiable, and u_λ and θ_λ are the Hölder functions in the variable λ.*

The proof of the theorem follows from the definition of hyperbolicity and Lemmas 1 and 2.

3. Change in the Type of the System of Equations as the Flow Evolves. We shall give an exact solution in which Eqs. (1.7) change type with time. We consider the solution

$$u=(x+C(\lambda))(t+C(\lambda)+a)^{-1},\qquad \theta=u^{-1}+(1-u)^{-1}\qquad (0<u<1), \tag{3.1}$$

where $C(\lambda)$ is an arbitrary function [$C'(\lambda)>0$, $C(0)=0$, and $C(1)=C_1$] and a is a positive constant. The inequalities $u_\lambda>0$ and $H=\Phi_\lambda=C'(x+C)^{-1}>0$, which ensure that the velocity profile is monotonic with depth and the change of (1.3) is reversible, are satisfied in the region $0<x<t+a$ ($t\geqslant 0$). The upper boundary of the channel and the flow rate are given by the functions

$$h(x)=\ln(1+C_1x^{-1}),\qquad Q(t)=\ln(1+C_1(t+a)^{-1}).$$

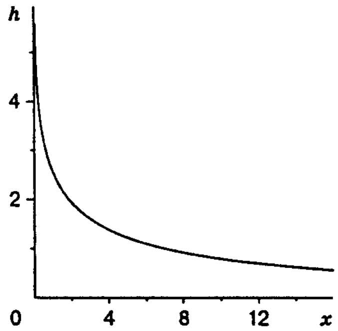

Fig. 2

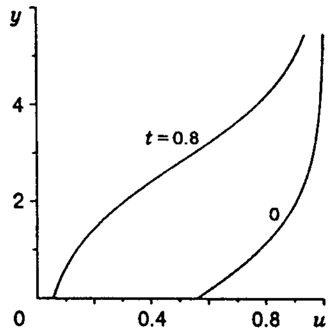

Fig. 3

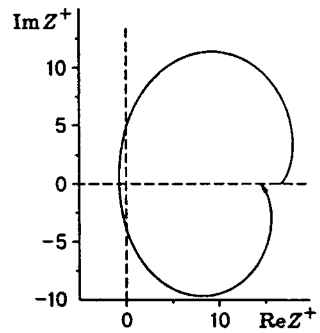

Fig. 4

The solution in Eulerian coordinates has the form

$$u = x\exp(y)(t + x(\exp(y) - 1) + a)^{-1},$$

$$v = -(\exp(y) - 1)(t + x(\exp(y) - 1) + a)^{-1}, \qquad p = -y + f(t),$$

where $f(t)$ is an arbitrary function. For definiteness, let $C_1 = 12$ and $a = 0.09$.

This solution describes a liquid flow in a convergent channel (Fig. 2). The horizontal velocity component u is greater than zero, and the liquid therefore flows in the positive direction of the x axis. For a fixed t and large values of x, the channel height and the vertical velocity component approach zero, while the horizontal component approaches infinity. With increasing time (for a fixed x), the flow slows down, since $u \to 0$ and $v \to 0$. In Fig. 3, we show velocity profiles for $x = 0.05$ at the times $t = 0$ and 0.8.

We verify the hyperbolicity conditions for system (1.7) using the functions $Z^{\pm}(u) = (u_1 - u)(u_0 - u)\chi^{\pm}(u)$. For solution (3.1), the functions $Z^{\pm}$ have the form

$$Z^{\pm}(u) = (u - u_0)[u_1^{-1} + (1 - u_1)^{-1}] + (u_1 - u)[u_0^{-1} + (1 - u_0)^{-1}]$$
$$-(u_1 - u)(u - u_0)[(1 - u)^{-1}((1 - u_1)^{-1} - (1 - u_0)^{-1}) + u^{-1}(u_0^{-1} - u_1^{-1})$$
$$+(1 - u)^{-2}\ln((1 - u_0)(u_1 - u)(1 - u_1)^{-1}(u - u_0)^{-1})$$
$$+u^{-2}\ln(u_1(u - u_0)u_0^{-1}(u_1 - u)^{-1})] \mp \pi i(u_1 - u)(u - u_0)((1 - u)^{-2} - u^{-2}).$$

It is seen that the imaginary part of $Z^{\pm}$ vanishes only for $u = u_0$, $1/2$, and u_1. If we have $0 < u < 1/2$ or $1/2 < u < 1$, then $\operatorname{Im} Z^{\pm}(u)$ does not change sign and conditions (2.10) are satisfied.

Let us test whether the hyperbolicity conditions for solution (3.1) are satisfied at the point $x = 0.05$ at the times $t = 0$ and 0.8. For $t = 0$, we have the index $æ = 0$ [conditions (2.10) are satisfied], since $u_0 = 0.55556 > 1/2$, $\operatorname{Im} Z^{\pm}$ does not change sign as u varies from u_0 to u_1, and the increment of the argument of the functions $Z^{\pm}(u)$ is zero. Moreover, at the initial time $t = 0$ the hyperbolicity conditions (2.10) are satisfied at any point x at which the solution is determined.

For $t = 0.8$, on the basis of Fig. 4 (the graph of the function Z^- is similar, but goes around in the opposite direction) we obtain $\Delta \arg Z^+(u) = 2\pi$ and $\Delta \arg Z^-(u) = -2\pi$, and hence $æ = 4\pi$. The hyperbolicity conditions are violated in this case, and there are complex characteristic roots for the solution under consideration. This example shows that as the flow evolves, system (1.7) can change type, and in a liquid flow in a narrow channel, instabilities may develop for certain distributions of the initial data.

4. Solutions of the Linearized Problem. We linearize Eqs. (1.7) for the solution $u = u^0(\lambda)$ ($u^0_\lambda \neq 0$) and $\theta = \theta^0(\lambda)$. For this we represent the functions u and θ as

$$u(t, x, \lambda) = u^0(\lambda) + \varepsilon u'(t, x, \lambda), \qquad \theta(t, x, \lambda) = \theta^0(\lambda) + \varepsilon\theta'(t, x, \lambda),$$

where $u'(t, x, \lambda)$ and $\theta'(t, x, \lambda)$ are the desired quantities (perturbations) and ε is a small parameter. The function $Q(t)$ giving the liquid flow rate in the channel is also linearized: $Q(t) = Q_0 + \varepsilon Q_1(t)$. We assume the function $Q_1(t)$ to be given. We omit the prime below. The system for determination of perturbations has the form

$$u_{\lambda t} + u^0 u_{\lambda x} + u^0_\lambda u_x = 0, \qquad \theta_t + u^0\theta_x = 0. \tag{4.1}$$

Here the functions u and u_x are expressed in terms of u^0, θ^0, u_λ, θ, and Q_1 by the equations

$$u = \Big(\int_0^1 u^0_\lambda \theta^0\,d\lambda\Big)^{-1}\Big[Q_1(t) + \int_0^1 u_\lambda\Big(\int_0^\lambda u^0_\nu\theta^0\,d\nu\Big)\,d\lambda - u^0_1\int_0^1 u_\lambda\theta^0\,d\lambda$$

$$+\int_0^1 u^0_\lambda\Big(\int_0^\lambda u_\nu\theta^0\,d\nu\Big)\,d\lambda - u^0_1\int_0^1 u^0_\lambda\theta\,d\lambda + \int_0^1 u^0_\lambda\Big(\int_0^\lambda u^0_\nu\theta\,d\nu\Big)\,d\lambda\Big] - \int_\lambda^1 u_\nu\,d\nu,$$

$$u_x = \Big(\int_0^1 u^0_\lambda\theta^0\,d\lambda\Big)^{-2}\Big[Q_0 + \int_0^1 u^0_\lambda\Big(\int_0^\lambda u^0_\nu\theta^0\,d\nu\Big)\,d\lambda\Big]\Big(\int_0^1 u_{\lambda x}\theta^0\,d\lambda + \int_0^1 u^0_\lambda\theta_x\,d\lambda\Big)$$

$$+\Big(\int_0^1 u^0_\lambda\theta^0 d\lambda\Big)^{-1}\Big[\int_0^1 u_{\lambda x}\Big(\int_0^\lambda u^0_\nu\theta^0\,d\nu\Big)d\lambda + \int_0^1 u^0_\lambda\Big(\int_0^\lambda u_{\nu x}\theta^0\,d\nu\Big)d\lambda + \int_0^1 u^0_\lambda\Big(\int_0^\lambda u^0_\nu\theta_x\,d\nu\Big)d\lambda\Big] - \int_\lambda^1 u_{\nu x}\,d\nu$$

obtained by linearizing (1.6) and (1.8). Equations (4.1) thus represent a linear system for determination of u_λ and θ.

Equations (4.1) [by analogy with the nonlinear system (1.7)] are equivalent to the equations for the characteristics

$$u^0(R_t + u^0R_x) + h_t + u^0h_x = 0, \qquad \theta_t + u^0\theta_x = 0, \qquad h_t = 0, \qquad u^0 \neq 0, \tag{4.2}$$

where

$$R = \int_0^1 [\theta^0(\nu)u_\nu + u^0_\nu\theta(\nu) - u^0_\nu\theta^0(\nu)(u(\nu) - u(\lambda))(u^0(\nu) - u^0(\lambda))^{-1}](u^0(\nu) - u^0(\lambda))^{-1}\,d\nu,$$

$$h = \int_0^1 (u^0_\lambda\theta + \theta^0 u_\lambda)\,d\lambda. \tag{4.3}$$

If $u^0(\lambda)$ vanishes, the linearized equations (2.9′) must be used. By integrating Eqs. (4.2), we obtain

$$R(t,x,\lambda) = f_1(x - tu^0(\lambda),\lambda) - (u^0(\lambda))^{-1}h(x), \qquad \theta(t,x,\lambda) = f_2(x - tu^0(\lambda),\lambda)$$

(the functions f_1 and f_2 are defined by the initial data). Suppose we have the initial data

$$u_\lambda(0,x,\lambda) = s(x,\lambda), \qquad \theta(0,x,\lambda) = f_2(x,\lambda).$$

Then f_1 is found by substituting the functions s and f_2 into (4.3) in place of u_λ and θ. To solve system (4.1), we must express u_λ in terms of the known quantities R, θ, h, u^0, and θ^0.

Proceeding by analogy with the proof of Lemma 2, we introduce the function $\psi(\lambda) = -\int_\lambda^1 u_\nu\,d\nu$ and represent Eqs. (4.3) in the form

$$\theta^0_1\psi(\lambda)(u^0_1 - u^0(\lambda))^{-1} + \theta^0_0(\psi_0 - \psi(\lambda))(u^0_0 - u^0(\lambda))^{-1} + \int_0^1 \theta^0_\nu(\psi(\nu) - \psi(\lambda))(u^0(\nu) - u^0(\lambda))^{-1}\,d\nu$$

$$= R(\lambda) - \int_0^1 \theta(\nu)u^0_\nu(u^0(\nu) - u^0(\lambda))^{-1}\,d\nu, \qquad \int_0^1 \theta^0\psi_\lambda\,d\lambda = h - \int_0^1 \theta u^0_\lambda\,d\lambda. \tag{4.3′}$$

We express ψ_0 by means of the second equation of (4.3′) and substitute it into the first, and after a transformation we obtain the equation

$$g(\lambda) = \psi(\lambda)(u_0^0 - u^0(\lambda))\Big[-\theta_1^0(u_1^0 - u^0(\lambda))^{-1} + \theta_0^0(u_0^0 - u^0(\lambda))^{-1} + \int_0^1 \theta_\nu^0(u^0(\nu) - u^0(\lambda))^{-1}\,d\nu\Big]$$

$$+ \int_0^1 \theta_\nu(u^0(\nu) - u_0^0)(u^0(\nu) - u^0(\lambda))^{-1}\psi(\nu)\,d\nu,$$

$$g(\lambda) = -h + \int_0^1 u_\lambda^0\theta\,d\lambda + (u_0^0 - u^0(\lambda))\Big[R(\lambda) - \int_0^1 u_\nu^0\theta(\nu)(u^0(\nu) - u^0(\lambda))^{-1}\,d\nu\Big],$$

which we reduce, by the change of the variable $\xi = u^0(\nu)$ [$\xi_0 = u_0^0$, $\xi_1 = u_1^0$, and $z = u(\lambda)$], to a singular integral equation similar to the characteristic equation [4]:

$$\tilde{\psi}(z)\tilde{K}(z)(\xi_1 - z)^{-1} - \Psi(z) = \tilde{g}(z).$$

Here

$$\Psi(z) = \int_{\xi_0}^{\xi_1} \tilde{\theta}_\xi(\xi_0 - \xi)(\xi - z)^{-1}\tilde{\psi}(\xi)\,d\xi;$$

$$\tilde{K}(z) = (\xi_0 - z)(\xi_1 - z)\Big[-\tilde{\theta}_1^0(\xi_1 - z)^{-1} + \tilde{\theta}_0^0(\xi_0 - z)^{-1} + \int_{\xi_0}^{\xi_1} \tilde{\theta}_\xi^0(\xi - z)^{-1}\,d\xi\Big].$$

As in Sec. 2, we use the notation $f(\lambda) = \tilde{f}(z)$. Solving the integral equation is reduced to solving the inhomogeneous conjugation problem

$$\Psi^+(z) = G(z)\Psi^-(z) + 2\pi i(\xi_0 - z)(\xi_1 - z)\tilde{\theta}_z^0\tilde{g}(z)/\tilde{K}^-(z), \qquad G = \tilde{K}^+/\tilde{K}^-.$$

We note that $\tilde{K}(z)$ satisfies the requirements of a canonical function and coincides, to within the cofactor, with the function $\tilde{\chi}(z)$, in terms of the limiting values of which we formulated the conditions for hyperbolicity of Eqs. (1.7). The index of the conjugation problem is therefore zero. In accordance with [4], we write the solution of the inhomogeneous conjugation problem, which vanishes at infinity, as follows:

$$\Psi(z) = \tilde{K}(z)\int_{\xi_0}^{\xi_1}(\xi_0 - \xi)(\xi_1 - \xi)\tilde{\theta}_\xi^0\tilde{g}(\xi)[\tilde{K}^+(\xi)\tilde{K}^-(\xi)(\xi - z)]^{-1}\,d\xi.$$

The solution $\tilde{\psi}(z)$ of the integral equation is

$$\tilde{\psi}(z) = (2\pi i(\xi_0 - z)\tilde{\theta}_z^0)^{-1}[\Psi^+(z) - \Psi^-(z)].$$

Making direct calculations and performing a reverse change of the variables, we find

$$\psi(\lambda) = (u_1^0 - u^0(\lambda))[K(\lambda)g(\lambda)(K^+(\lambda)K^-(\lambda))^{-1} + N(\lambda)],$$

$$N(\lambda) = \int_0^1 (u_0^0 - u^0(\nu))(u_1^0 - u^0(\nu))\theta_\nu^0 g(\nu)[(K^+K^-)(\nu)(u^0(\nu) - u^0(\lambda))]^{-1}\,d\nu.$$

The unknown function u_λ is now determined by differentiating ψ with respect to the variable λ. We thus obtain the solution of the Cauchy problem for system (4.1).

For system (1.7), we can construct an example of an ill-posed Cauchy problem if, for the solution $u = u^0(\lambda)$ and $\theta = \theta^0(\lambda)$ considered, there are complex roots of Eq. (2.8). System (4.1) has the solution

$$u = \varphi_1(\lambda)\exp(il(x - kt)), \qquad \theta = \varphi_2(\lambda),$$

where k is a complex root ($\operatorname{Im} k > 0$) of the characteristic equation (2.8). The function $u(0, x, \lambda)$ is finite as $l \to \infty$, but $u(t, x, \lambda)$ ($t > 0$) becomes infinite. The lack of a continuous dependence of the solution on the initial data indicates that the Cauchy problem is ill-posed with violation of conditions (2.10).

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